5 Banach Algebras

5.1 Invertibility and the Spectrum

Suppose X is a Banach space. Then we are often interested in (continuous) operators on this space, i.e, elements of the space $\mathrm{CL}(X, X)$. We have already seen that this is again a Banach space. However, operators can also be composed with each other, which gives us more structure, namely that of an algebra. It is often useful to study this abstractly, i.e., forgetting about the original space on which the operators X act. This leads us to the concept of a *Banach algebra*.

Definition 5.1. Let A be an algebra over $\mathbb{K}$. A is called a *commutative* algebra iff $a \cdot b = b \cdot a$ for all $a, b \in A$. An element $e \in A$ is called a *unit* iff $e \cdot a = a \cdot e = a$ for all $a \in A$ and $e \neq 0$. Iff A is equipped with a unit it is called a *unital* algebra. Assume now A to be unital and consider $a \in A$. Then, $b \in A$ is called an *inverse* of a iff $b \cdot a = a \cdot b = e$. An element $a \in A$ possessing an inverse is called *invertible*.

It is immediately verified that a unit and an inverse are unique. In the following of this section we work exclusively over the field $\mathbb{C}$ of complex numbers.

Definition 5.2 (Banach Algebra)**.** A is called a *Banach algebra* iff it is a complete normable topological algebra.

Proposition 5.3. *Let A be a complete normable tvs and an algebra. Then, A is a Banach algebra iff there exists a compatible norm on A such that $\|a \cdot b\| \le \|a\| \cdot \|b\|$ for all $a, b \in A$. Moreover, if A is unital then it is a Banach algebra iff there exists a compatible norm that satisfies in addition $\|e\| = 1$.*

Proof. Suppose that A admits a norm generating the topology and satisfying $\|a \cdot b\| \le \|a\| \cdot \|b\|$ for all $a, b \in A$. Fix $a, b \in A$ and let $\epsilon > 0$. Choose $\delta > 0$ such that

$$(\|a\| + \|b\|)\delta + \delta^2 < \epsilon.$$

Then,

$$\begin{aligned}\|(a+x)\cdot(b+y) - a\cdot b\| = \|a\cdot y + x\cdot b + x\cdot y\| &\le \|a\cdot y\| + \|x\cdot b\| + \|x\cdot y\| \\ &\le \|a\|\cdot\|y\| + \|x\|\cdot\|b\| + \|x\|\cdot\|y\| < \epsilon\end{aligned}$$

if $x, y \in B_\delta(0)$, showing continuity of multiplication.

Now suppose that A is a Banach algebra. Let $\|\cdot\|'$ be a norm generating the topology. By continuity there exists $\delta > 0$ such that $\|a \cdot b\|' \leq 1$ for all $a, b \in B_\delta(0)$. But this implies $\|a \cdot b\|' \leq \delta^{-2}\|a\|' \cdot \|b\|'$ for all $a, b \in A$. It is then easy to see that $\|a\| := \delta^{-2}\|a\|'$ for all $a \in A$ defines a norm that is topologically equivalent and satisfies $\|a \cdot b\| \leq \|a\| \cdot \|b\|$ for all $a, b \in A$.

Now suppose that A is a unital Banach algebra. Let $\|\cdot\|'$ be a norm generating the topology. As we have just seen there exists a constant $c > 0$ such that $\|a \cdot b\|' \leq c\|a\|' \cdot \|b\|'$ for all $a, b \in A$. We claim that

$$\|a\| := \sup_{\|b\|' \leq 1} \|a \cdot b\|' \quad \forall a \in A$$

defines a topologically equivalent norm with the desired properties. It is easy to see that $\|\cdot\|$ is a seminorm. Now note that

$$\|a\| = \sup_{\|b\|' \leq 1} \|a \cdot b\|' \leq c \sup_{\|b\|' \leq 1} \|a\|' \cdot \|b\|' = c\|a\|' \quad \forall a \in A.$$

On the other hand we have

$$\|a\| = \sup_{\|b\|' \leq 1} \|a \cdot b\|' \geq \frac{\|a \cdot e\|'}{\|e\|'} = \frac{\|a\|'}{\|e\|'} \quad \forall a \in A.$$

This shows that $\|\cdot\|$ is indeed a norm and generates the same topology as $\|\cdot\|'$. The proof of the property $\|a \cdot b\| \leq \|a\| \cdot \|b\|$ for all $a, b \in A$ now proceeds as in Exercise 24. Finally, it is easy to see that $\|e\| = 1$. □

We have already seen the prototypical example of a Banach algebra in Exercise 24: The algebra of continuous linear operators $\mathrm{CL}(X, X)$ on a Banach space X.

Exercise 26. Let T be a compact topological space. Show that $\mathrm{C}(T, \mathbb{C})$ with the supremum norm is a unital commutative Banach algebra.

Exercise 27. Consider the space $l^1(\mathbb{Z})$, i.e., the space of complex sequences $\{a_n\}_{n \in \mathbb{Z}}$ with $\|a\| := \sum_{n \in \mathbb{Z}} |a_n| < \infty$. 1. Show that this is a Banach space. 2. Define a multiplication by convolution, i.e., $(a \star b)_n := \sum_{k \in \mathbb{Z}} a_k b_{n-k}$. Show that this is well defined and yields a commutative Banach algebra.

Proposition 5.4. *Let A be a unital Banach algebra and $a \in A$. If $\|e - a\| < 1$ then a is invertible. Moreover, in this case*

$$a^{-1} = \sum_{n=0}^{\infty} (e - a)^n \quad \textit{and} \quad \|a^{-1}\| \leq \frac{1}{1 - \|e - a\|}.$$

Proof. **Exercise.** □

Proposition 5.5. *Let A be a unital Banach algebra. Denote the subset of invertible elements of A by I_A. Then, I_A is open. Moreover, the map $I_A \to I_A : a \mapsto a^{-1}$ is continuous.*

Proof. Consider an invertible element $a \in I_A$ and choose $\epsilon > 0$. Set

$$\delta := \min\left\{\frac{1}{2}\|a^{-1}\|^{-1}, \frac{1}{2}\epsilon\|a^{-1}\|^{-2}\right\}.$$

Take $b \in B_\delta(a)$. Then $b = a(e + a^{-1}(b - a))$. But

$$\|a^{-1}(b-a)\| \le \|a^{-1}\|\|b-a\| < \|a^{-1}\|\delta \le \frac{1}{2}.$$

So by Proposition 5.4 the element $e + a^{-1}(b-a)$ is invertible. Consequently, b is a product of invertible elements and hence itself invertible. Therefore, $B_\delta(a) \subseteq I_A$ and I_A is open. Furthermore, using the same inequality we find by Proposition 5.4 that

$$\|(e + a^{-1}(b-a))^{-1}\| \le \frac{1}{1 - \|a^{-1}(b-a)\|} < 2.$$

This implies

$$\|b^{-1}\| \le \|a^{-1}\|\|(e + a^{-1}(b-a))^{-1}\| < 2\|a^{-1}\|.$$

Hence,

$$\|a^{-1} - b^{-1}\| = \|a^{-1}(b-a)b^{-1}\| \le \|a^{-1}\|\|b^{-1}\|\|b-a\| < 2\|a^{-1}\|^2\delta \le \epsilon.$$

This shows the continuity of the inversion map, completing the proof. □

Definition 5.6. Let A be a unital Banach algebra and $a \in A$. Then, the set $\sigma_A(a) := \{\lambda \in \mathbb{C} : \lambda e - a \text{ not invertible}\}$ is called the *spectrum* of a.

Proposition 5.7. *Let A be a unital Banach algebra and $a \in A$. Then the spectrum $\sigma_A(a)$ of a is a compact subset of $\mathbb{C}$. Moreover, $|\lambda| \le \|a\|$ if $\lambda \in \sigma_A(a)$.*

Proof. Consider $\lambda \in \mathbb{C}$ such that $|\lambda| > \|a\|$. Then, $\|\lambda^{-1}a\| = |\lambda^{-1}|\|a\| < 1$. So, $e - \lambda^{-1}a$ is invertible by Proposition 5.4. Equivalently, $\lambda e - a$ is invertible and hence $\lambda \notin \sigma_A(a)$. This proves the second statement and also implies that $\sigma_A(a)$ is bounded.

It remains to show that $\sigma_A(a)$ is closed. Take $\lambda \notin \sigma_A(a)$. Set $\epsilon := \|(\lambda e - a)^{-1}\|^{-1}$. We claim that for all $\lambda' \in B_\epsilon(\lambda)$ the element $\lambda' e - a$ is invertible. Note that $\|(\lambda - \lambda')(\lambda e - a)^{-1}\| = |\lambda - \lambda'|\|(\lambda e - a)^{-1}\| < \epsilon\|(\lambda e - a)^{-1}\| = 1$. So by Proposition 5.4 the element $e - (\lambda - \lambda')(\lambda e - a)^{-1}$ is invertible. But the product of invertible elements is invertible and so is hence $\lambda' e - a = (\lambda e - a)(e - (\lambda - \lambda')(\lambda e - a)^{-1})$, proving the claim. Thus, $\mathbb{C} \setminus \sigma_A(a)$ is open and $\sigma_A(a)$ is closed, completing the proof. □

Lemma 5.8. *Let A be a unital algebra and $a, b \in A$. Suppose that $a \cdot b$ and $b \cdot a$ are invertible. Then, a and b are separately invertible.*

Proof. **Exercise.** □

Theorem 5.9 (Spectral Mapping Theorem)**.** *Let A be a unital Banach algebra, p a complex polynomial in one variable and $a \in A$. Then, $\sigma_A(p(a)) = p(\sigma_A(a))$.*

Proof. If p is a constant the statement is trivially satisfied. We thus assume in the following that p has degree at least 1.

We first prove that $p(\sigma_A(a)) \subseteq \sigma_A(p(a))$. Let $\lambda \in \mathbb{C}$. Then the polynomial in t given by $p(t) - p(\lambda)$ can be decomposed as $p(t) - p(\lambda) = q(t)(t - \lambda)$ for some polynomial q. In particular, $p(a) - p(\lambda) = q(a)(a - \lambda)$ in A. Suppose $p(\lambda) \notin \sigma_A(p(a))$. Then the left hand side is invertible and so must be the right hand side. By Lemma 5.8 each of the factors must be invertible. In particular, $a - \lambda$ is invertible and so $\lambda \notin \sigma_A(a)$. We have thus shown that $\lambda \in \sigma_A(a)$ implies $p(\lambda) \in \sigma_A(p(a))$.

We proceed to prove that $\sigma_A(p(a)) \subseteq p(\sigma_A(a))$. Let $\mu \in \mathbb{C}$ and factorize the polynomial in t given by $p(t) - \mu$, i.e., $p(t) - \mu = c(t - \gamma_1) \cdots (t - \gamma_n)$, where $c \neq 0$. We apply this to a to get $p(a) - \mu = c(a - \gamma_1) \cdots (a - \gamma_n)$. Now if $\mu \in \sigma_A(p(a))$, then the left hand side is not invertible. Hence, at least one factor $a - \gamma_k$ must be non-invertible on the right hand side. So, $\gamma_k \in \sigma_A(a)$ and also $\mu = p(\gamma_k)$. Thus, $\mu \in p(\sigma_A(a))$. This completes the proof. □

Definition 5.10. Let A be a Banach algebra and $a \in A$. We define the *spectral radius* of a as

$$r_A(a) := \inf_{n \in \mathbb{N}} \|a^n\|^{1/n}.$$

Lemma 5.11. *Let $\{c_n\}_{n \in \mathbb{N}}$ be a sequence of non-negative real numbers such that $c_{n+m} \leq c_n c_m$ for all $n, m \in \mathbb{N}$. Then $\{c_n^{1/n}\}_{n \in \mathbb{N}}$ converges to $\inf_{n \in \mathbb{N}} c_n^{1/n}$.*

Proof. Define $c_0 := 1$. For fixed m decompose any positive integer $n = k(n)m + r(n)$ such that $r(n), k(n) \in \mathbb{N}_0$ and $r(n) < m$. Then,

$$c_n^{1/n} \le c_{k(n)m}^{1/n} c_{r(n)}^{1/n} \le c_m^{k(n)/n} c_{r(n)}^{1/n}.$$

Since $r(n)$ is bounded and $k(n)/n$ converges to $1/m$ for large n the right hand side tends to $c_m^{1/m}$ for large n. This implies,

$$\limsup_{n\to\infty} c_n^{1/n} \le c_m^{1/m}.$$

Since m was arbitrary we conclude,

$$\limsup_{n\to\infty} c_n^{1/n} \le \inf_{n\in\mathbb{N}} c_n^{1/n} \le \liminf_{n\to\infty} c_n^{1/n}.$$

This completes the proof. □

Proposition 5.12. *Let A be a Banach algebra and $a \in A$. Then,*

$$\lim_{n\to\infty} \|a^n\|^{1/n} \quad \textit{exists and is equal to} \quad \inf_{n\in\mathbb{N}} \|a^n\|^{1/n}.$$

Proof. If a is nilpotent (i.e., $a^n = 0$ for some n) the statement is trivial. Assume otherwise and set $c_n := a^n$. Applying Lemma 5.11 yields the result. □

Lemma 5.13. *Let A be a unital Banach algebra, $a \in A$ and $r > 0$ such that $\sigma_A(a) \subset B_r(0)$. Let γ be the path following the boundary of the circle of radius r around the origin in counter-clockwise direction. Then,*

$$a^n = \frac{1}{2\pi\mathrm{i}} \int_\gamma z^n (z-a)^{-1} \mathrm{d}z.$$

Note that if $A = \mathbb{C}$ this is merely a consequence of the Cauchy integral formula. The present case is a generalization to Banach algebras.

Theorem 5.14. *Let A be a unital Banach algebra and $a \in A$. Then*

$$r_A(a) = \sup_{\lambda\in\sigma_A(a)} |\lambda|.$$

In particular, $\sigma_A(a) \neq \emptyset$.

Proof. Choose $\lambda \in \mathbb{C}$ such that $|\lambda| > r_A(a)$. Then there exists $n \in \mathbb{N}$ such that $|\lambda| > \|a^n\|^{1/n}$ and hence $|\lambda^n| > \|a^n\|$. By Proposition 5.7 we know that $\lambda^n \notin \sigma_A(a^n)$. By Theorem 5.9 with $p(t) = t^n$ this implies $\lambda \notin \sigma_A(a)$. This shows $|\lambda| \le r_A(a)$ for all $\lambda \in \sigma_A(a)$.

Choose now $r > 0$ so that $\sigma_A(a)$ is contained completely in the circle of radius r around the origin in the complex plane. By Lemma 5.13 we get

$$\|a^n\| = \frac{1}{2\pi}\left\|\int_\gamma z^n(z-a)^{-1}\mathrm{d}z\right\| \le \frac{1}{2\pi} r^n m(r) l(\gamma) = r^{n+1} m(r), \qquad (2)$$

where we have defined $m(r) := \max_{|z|=r} \|(z-a)^{-1}\|$. Also, $l(\gamma)$ denotes the length of the path γ. Note that $(z-a)^{-1}$ is a continuous function on γ since inversion in A is continuous by Proposition 5.5, so the maximum exists. Suppose now that $\sigma_A(a) = \emptyset$. Then, a is invertible, $(z-a)^{-1}$ is a continuous function on all of $\mathbb{C}$ and $m(r)$ converges for $r \to 0$ to $m(0) = \|a^{-1}\|$. Taking $n = 1$ in the inequality we obtain $\|a\| \le r^2 m(r)$ which implies $\|a\| = 0$ since r is now arbitrary. Hence, $a = 0$ which is a contradiction with the invertibility of a. This shows that $\sigma_A(a) \neq \emptyset$.

From the inequality (2) we infer,

$$\|a^n\|^{1/n} \le r^{(n+1)/n} m^{1/n}.$$

Taking the limit $n \to \infty$ on both sides (the existence of the limit on the left hand side is ensured by Proposition 5.12) we get $r_A(a) \le r$. But in particular, we may choose $r = \epsilon + \sup_{\lambda \in \sigma_A(a)} |\lambda|$ for arbitrary $\epsilon > 0$. This completes the proof. □

Theorem 5.15 (Gelfand-Mazur)**.** *Let A be a unital Banach algebra such that all its non-zero elements are invertible. Then A is isomorphic to $\mathbb{C}$ as a Banach algebra.*

Proof. **Exercise.** □

5.2 The Gelfand Transform

Suppose we have some topological space T. Then, this space gives rise to a commutative algebra, namely the algebra of continuous functions on T (with complex values say). A natural question arises thus: If we are given a commutative algebra, is the algebra of continuous functions on some topological space? We might refine the question, considering more specific spaces such as Hausdorff spaces, manifolds etc. On the other hand we could also consider other classes of functions, e.g., differentiable ones etc. The

Gelfand transform goes towards answering this question in the context of unital commutative Banach algebras on the one hand and compact Hausdorff spaces on the other.

5.2.1 Ideals

Definition 5.16. Let A be an algebra. An *ideal* in A is a vector subspace J of A such that $aJ \subseteq J$ and $Ja \subseteq J$ for all $a \in A$. An ideal is called *proper* iff it is not equal to A. An ideal is called *maximal* iff it is proper and it is not contained in any other proper ideal.

The special significance of maximal ideals for our present purposes is revealed by the following Exercise. This also provides a preview of what we are going to show.

Exercise 28. Consider the Banach algebra $\mathrm{C}(T, \mathbb{C})$ of Exercise 26. Assume in addition that T is Hausdorff. 1. Show that for any non-empty subset U of T the set $\{f \in \mathrm{C}(T, \mathbb{C}) : f(U) = 0\}$ forms a proper closed ideal. 2. Show that the maximal ideals are in one-to-one correspondence to points of T.

Proposition 5.17. *Let A be a Banach algebra. Then, the closure of an ideal is an ideal.*

Proof. Let J be an ideal. We already know that $\overline{J}$ is a vector subspace. It remains to show the property $a\overline{J} \subseteq \overline{J}$ and $\overline{J}a \subseteq \overline{J}$ for all $a \in A$. Consider $b \in \overline{J}$. Then, there is a sequence $\{b_n\}_{n\in\mathbb{N}}$ with $b_n \in J$ converging to b. Take now $a \in A$ and consider the sequences $\{ab_n\}_{n\in\mathbb{N}}$ and $\{b_n a\}_{n\in\mathbb{N}}$. Since J is an ideal the elements of these sequences are all in J. And since multiplication by a fixed element is continuous the sequences converge to ab and ba respectively. So $ba \in \overline{J}$ and $ab \in \overline{J}$. This completes the proof. □

Proposition 5.18. *Let A be a unital Banach algebra.*

1. *If $a \in A$ is invertible it is not contained in any proper ideal.*
2. *Maximal ideals are closed.*
3. *Any proper ideal is contained in a maximal ideal.*

Proof. Suppose J is an ideal containing an invertible element $a \in A$. Then, $a^{-1}a = e \in J$ and thus $J = A$. This proves 1. Suppose J is a proper ideal. Then, $\overline{J}$ is an ideal by Proposition 5.17. On the other hand, by 1. the intersection of the set I_A of invertible elements of A with J is empty. But by Proposition 5.5 this set is open, so $I_A \cap \overline{J} = \emptyset$. Since $e \in I_A$, $\overline{J} \neq A$, i.e., $\overline{J}$ is proper. So we get an inclusion of proper ideals, $J \subseteq \overline{J}$. If J is maximal we must therefore have $J = \overline{J}$. This proves 2. The proof of 3 is a standard application of Zorn's Lemma. □

Proposition 5.19. *Let A be a Banach algebra and J a closed proper ideal. Then, A/J is a Banach algebra with the quotient norm. If A is unital then so is A/J. If A is commutative then so is A/J.*

Proof. **Exercise.** □

Definition 5.20. Let A be a Banach algebra. The set of maximal ideals of A is called the *maximal ideal space* and denoted by M_A. The set of maximal ideals with codimension 1 is denoted by M_A^1.

Proposition 5.21. *Let A be a commutative unital Banach algebra. Then, maximal ideals have codimension 1. In particular, $M_A = M_A^1$.*

Proof. Let J be a maximal ideal. By Proposition 5.18.2, J is closed. Hence, by Proposition 5.19, A/J is a unital commutative Banach algebra. We show that every non-zero element of A/J is invertible. For $a \in A \setminus J$ set $J_a := \{ab + c : b \in A \text{ and } c \in J\}$. It is easy to see that J_a is an ideal and $J \subset J_a$ as well as $J_a \neq J$. Since J is maximal we find $J_a = A$. But his means there is a $b \in A$ such that $[a][b] = [e]$ in A/J, i.e., $[a]$ is invertible in A/J. But every non-zero element of A/J arises as $[a]$ with $a \in A \setminus J$, so they are all invertible. By the Theorem 5.15 of Gelfand-Mazur we find that A/J is isomorphic to $\mathbb{C}$ and hence 1-dimensional. So, J must have codimension 1. □

5.2.2 Characters

Definition 5.22. Let A be a Banach algebra. An algebra homomorphism $\phi : A \to \mathbb{C}$ is called a *character* of A.

Proposition 5.23. *Let A be a Banach algebra. Then, any character $\phi : A \to \mathbb{C}$ is continuous. Moreover, $\|\phi\| \leq 1$. If A is also unital and $\phi \neq 0$ then $\phi(e) = 1$ and $\|\phi\| = 1$.*

Proof. Consider an algebra homomorphism $\phi : A \to \mathbb{C}$. Suppose $|\phi(a)| > \|a\|$ for some $a \in A$. Then we can find $\lambda \in \mathbb{C}$ such that $\phi(\lambda a) = 1$ while $\|\lambda a\| < 1$. Set $b := \sum_{n=1}^{\infty} (\lambda a)^n$. Then $b = \lambda a + \lambda a b$ and we obtain the contradiction $\phi(b) = \phi(\lambda a) + \phi(\lambda a)\phi(b) = 1 + \phi(b)$. Thus, $|\phi(a)| \leq \|a\|$ for all $a \in A$ and ϕ must be continuous. Also, $\|\phi\| \leq 1$.

Now assume in addition that A is unital and $\phi \neq 0$. Then there exists $a \in A$ such that $\phi(a) \neq 0$. We deduce $\phi(e) = 1$ since $\phi(a) = \phi(ea) = \phi(e)\phi(a)$ and thus $\|\phi\| \geq 1$. □

Definition 5.24. Let A be a Banach algebra. The set of non-zero characters on A is called the *character space* or *Gelfand space* of A, denoted by Γ_A. We view Γ_A as a subset of A^*, but equipped with the weak* topology. Define the map $A \to \mathrm{C}(\Gamma_A, \mathbb{C})$ given by $a \mapsto \hat{a}$ where $\hat{a}(\phi) := \phi(a)$. This map is called the *Gelfand transform*.

Proposition 5.25. *Let A be a unital Banach algebra. Then, Γ_A is a compact Hausdorff space.*

Proof. Since A is Hausdorff with the weak* topology so is its subset Γ_A. Let $\phi \in \Gamma_A$. By Proposition 5.23, ϕ is contained in the unit ball $\overline{B_1(0)} \subset A^*$. But by Corollary 4.22, $\overline{B_1(0)}$ is compact in in the weak* topology so Γ_A is relatively compact. It remains to show that Γ_A is closed in weak* topology. Suppose $\phi \in \overline{\Gamma_A}$. Pick two arbitrary elements $a, b \in A$. We know that the Gelfand transforms $\hat{a}, \hat{b}$ are continuous functions on $\tilde{A}$. Hence, choosing an arbitrary $\epsilon > 0$ we can find $\phi' \in \Gamma_A$ such that $|\phi'(a) - \phi(a)| < \epsilon$ and $|\phi'(b) - \phi(b)| < \epsilon$ and $|\phi'(ab) - \phi(ab)| < \epsilon$. **Exercise.**Explain! Then, $|\phi'(a)\phi'(b) - \phi(a)\phi(b)| < \epsilon(|\phi(a)| + |\phi(b)| + \epsilon)$. But, ϕ' is a character, so $\phi'(a)\phi'(b) = \phi'(ab)$. Thus, $|\phi(a)\phi(b) - \phi(ab)| < \epsilon(1 + |\phi(a)| + |\phi(b)| + \epsilon)$. Since ϵ was arbitrary we conclude that $\phi(a)\phi(b) = \phi(ab)$. This argument holds for any a, b so ϕ is a character. We have thus shown that either $\overline{\Gamma_A} = \Gamma_A$ or $\overline{\Gamma_A} = \Gamma_A \cup \{0\}$. To exclude the second possibility we need the unitality of A. Consider the subset $E := \{\phi \in \tilde{A} : \phi(e) = 1\} \subset A^*$. This subset is closed since it is the preimage of the closed set $\{1\} \subset \mathbb{C}$ under the Gelfand transform $\hat{e}$ of the unit e of A. Now, $\Gamma_A \subseteq E$, but $\{0\} \notin E$, so $\{0\} \notin \overline{\Gamma_A}$. □

We are now ready to link the character space with the maximal ideal space introduced earlier. They are (essentially) the same!

Theorem 5.26. *Let A be a Banach algebra. There is a natural map $\gamma : \Gamma_A \to M_A^1$ given by $\phi \mapsto \ker \phi$. If A is unital, this map is bijective.*

Proof. Consider $\phi \in \Gamma_A$. Suppose $a \in \ker \phi$. Then, for any $b \in A$ we have $ab \in \ker \phi$ and $ba \in \ker \phi$ since $\phi(ab) = \phi(a)\phi(b) = 0$ and $\phi(ba) = \phi(b)\phi(a) = 0$. Thus, $\ker \phi$ is an ideal. It is proper since $\phi \neq 0$. Now choose $a \in A$ such that $\phi(a) \neq 0$. For arbitrary $b \in A$ there is then a $\lambda \in \mathbb{C}$ such that $\phi(b) = \phi(\lambda a)$, i.e., $\phi(b - \lambda a) = 0$ and $b - \lambda a \in \ker \phi$. In particular,

$b \in \lambda a + \ker \phi$. So $\ker \phi$ has codimension 1 in A and must be maximal. This shows that γ is well defined.

Suppose now that A is unital and that J is a maximal ideal of codimension 1. Note that we can write any element a of A uniquely as $a = \lambda e + b$ where $\lambda \in \mathbb{C}$ and $b \in J$. In order for $J = \ker \phi$ for some $\phi \in \Gamma_A$ we must then have $\phi(\lambda e + b) = \lambda \phi(e) + \phi(b) = \lambda$. This determines ϕ uniquely. Hence, γ is injective. On the other hand, this formula defines a non-zero linear map $\phi : A \to \mathbb{C}$. It is easily checked that it is multiplicative and thus a character. Hence, γ is surjective. □

Proposition 5.27. *Let A be a unital Banach algebra and $a \in A$. Then, $\{\phi(a) : \phi \in \Gamma_A\} \subseteq \sigma_A(a)$. If A is commutative, then even $\{\phi(a) : \phi \in \Gamma_A\} = \sigma_A(a)$. In particular, $\Gamma_A \neq 0$.*

Proof. Suppose $\lambda = \phi(a)$ for some $\phi \in \Gamma_A$. Then, $\phi(\lambda e - a) = 0$, i.e., $\lambda e - a \in \ker \phi$. But by Theorem 5.26, $\ker \phi$ is a maximal ideal which by Proposition 5.18.1 cannot contain an invertible element. So $\lambda e - a$ is not invertible and $\lambda \in \sigma_A(a)$. This proves the first statement.

Suppose now that A is commutative and let $\lambda \in \sigma_A(a)$. Define $J := \{b(\lambda e - a) : b \in A\}$. It is easy to see that J defines an ideal. It is proper, since $\lambda e - a$ is not invertible. So, by Proposition 5.18.3 it is contained in a maximal ideal J'. This maximal ideal has codimension 1 by Proposition 5.21 and induces by Theorem 5.26 a non-zero character ϕ with $\ker \phi = J'$. Hence, $\phi(\lambda e - a) = 0$ and $\phi(a) = \lambda$. This completes the proof. □

When Γ_A is compact, then the set of continuous functions of Γ_A forms a unital commutative Banach algebra by Exercise 26. We then have the following Theorem.

Theorem 5.28 (Gelfand Representation Theorem)**.** *Let A be a unital Banach algebra. The Gelfand transform $A \to \mathrm{C}(\Gamma_A, \mathbb{C})$ is a continuous unital algebra homomorphism. The image of A under the Gelfand transform, denoted $\hat{A}$, is a normed subalgebra of $\mathrm{C}(\Gamma_A, \mathbb{C})$. Moreover, $\|\hat{a}\| \le r_A(a) \le \|a\|$ and $\sigma_{\hat{A}}(\hat{a}) \subseteq \sigma_A(a)$ for all $a \in A$. If A is commutative we have the sharper statements $\|\hat{a}\| = r_A(a)$ and $\sigma_{\hat{A}}(\hat{a}) = \sigma_A(a)$.*

Proof. The property of being a unital algebra homomorphism is clear. For $a \in A$ we have $\|\hat{a}\| = \sup_{\phi \in \Gamma_A} |\phi(a)|$. By Proposition 5.27 combined with Theorem 5.14 we then find $\|\hat{a}\| \le r_A(a)$ and in the commutative case $\|\hat{a}\| = r_A(a)$. On the other hand Proposition 5.7 combined with Theorem 5.14 implies $r_A(a) \le \|a\|$. Thus, the Gelfand transform is bounded by 1 and hence continuous. Since the Gelfand transform is a unital algebra homomorphism, invertible elements are mapped to invertible elements, so $\sigma_{\hat{A}}(\hat{a}) \subseteq \sigma_A(a)$. Let $a \in A$ and consider $\lambda \in \mathbb{C}$. If $\phi(a) = \lambda$ for some $\phi \in \Gamma_A$ then $\lambda\hat{e} - \hat{a}$ vanishes on this ϕ and cannot be invertible in $\tilde{A}$, i.e., $\lambda \in \sigma_{\hat{A}}(\hat{a})$. Using Proposition 5.27 we conclude $\sigma_{\hat{A}}(\hat{a}) \supseteq \sigma_A(a)$ in the commutative case. □

Proposition 5.29. *Let A be a unital commutative Banach algebra. Suppose that $\|a^2\| = \|a\|^2$ for all $a \in A$. Then, the Gelfand transform $A \to \mathrm{C}(\Gamma_A, \mathbb{C})$ is isometric. In particular, it is injective and its image $\hat{A}$ is a Banach algebra.*

Proof. Under the assumption $\lim_{n\to\infty} \|a^n\|^{1/n}$, which exists by Proposition 5.12, is equal to $\|a\|$ for all $a \in A$. By the same Proposition then $r_A(a) = \|a\|$. So by Theorem 5.28, $\|\hat{a}\| = r_A(a) = \|a\|$. Isometry implies of course injectivity. Moreover, it implies that the image is complete since the domain is complete. So $\hat{A}$ is a Banach algebra. □

Exercise 29. Let $A = \mathrm{C}(T, \mathbb{C})$ be the Banach algebra of Exercises 26 and 28. Show that $\Gamma_A = T$ as topological spaces in a natural way and that the Gelfand transform is the identity under this identification.